Entrance Examination - Mathematics

Name and Surname – fill in the field	*Application No.*	*Test Sheet No.*
		4

Sets, relations, functions, logic

1 Which of the following propositional formulas is semantically equivalent to the formula $A \Rightarrow \neg(B \wedge \neg C)$?
(Symbols A, B, and C denote distinct propositional variables.)

A $A \vee (\neg B \vee C)$

B $\neg A \vee (B \vee \neg C)$

C $A \vee (B \vee C)$

***D** $\neg A \vee (\neg B \vee C)$

E $\neg A \wedge (\neg B \vee C)$

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2 Consider the following unary functions on the set of rational numbers: $f(x) = x + 1$, $g(x) = 3x$, $h(x) = x^2$. Define the function k as $f \circ g \circ h \circ f$, where $\circ$ denotes the composition of functions. Which of the following values is equal to $k(1)$?

A 38
B 37
***C** 13
D 15
E 17

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3 Consider the equivalence relation on the set of integers such that a number a is related to a number b if and only if $(|a| = |b| \wedge |a| \leq 2) \vee (|a| > 2 \wedge |b| > 2)$. How many equivalence classes does this relation have?

A 2
B 5
C infinitely many
D 6
***E** 4

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4 Let A and B be arbitrary sets, each containing **three elements**. What is the number of maps $f\colon A \to B$ that are bijective?

A 3
B 1
C 9
***D** 6
E 4

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5 Suppose that all variables are interpreted as elements of a set A and the binary predicate symbol R is interpreted as an ordering relation $\leq$ on the set A. Which of the following predicate formulas is true if and only if the relation $\leq$ has at least one maximal element?

A $\exists x\, \forall y\, (R(y, x))$

B $\exists x\, \forall y\, (x = y \vee \neg R(y, x))$

C $\forall x\, \exists y\, (x = y \vee \neg R(x, y))$

***D** $\exists x\, \forall y\, (x = y \vee \neg R(x, y))$

E $\forall x\, \exists y\, (x = y \vee \neg R(y, x))$

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6 Let A and B be arbitrary finite sets. Which of the following claims is generally valid? (Here $|A|$ denotes the number of elements in the set A and $A \times B$ denotes the Cartesian product of sets A and B.)

A $|A| \leq |B| \Rightarrow A \subseteq B$

B $|A \cup B| = |A| + |B|$

***C** $|A \times B| = |A| \cdot |B|$

D $A \neq B \Rightarrow |A| \neq |B|$

E $|A| < |A \cup B|$

Linear algebra

7 Consider the following matrix and its inverse
$$A = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 2 & 5 \\ 4 & 1 & 2 \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} a_{11} & X & a_{13} \\ a_{21} & Y & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$
Compute the value $X + Y$.

A 2

B 0

C The matrix A^{-1} does not exist.

***D** -2

E -4

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8 Compute $\begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 & 6 & 1 \end{pmatrix}$.

***A** $\begin{pmatrix} -6 & 18 & 3 \\ -2 & 6 & 1 \\ -4 & 12 & 2 \end{pmatrix}$

B (5)

C (2)

D The product of the given matrices is not defined.

E 2

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9 Consider the following system of equations over $\mathbb{R}$:
$$\begin{aligned} x + 3y + z &= 5, \\ 3x - 2y - 4z &= 1, \\ 2x + 17y + 9z &= 26. \end{aligned}$$
Which of the following statements is true?

A The system has exactly one solution.

B The system has infinitely many solutions and all solutions form a line in $\mathbb{R}^3$.

C The system has infinitely many solutions and all solutions form a plane in $\mathbb{R}^3$.

D All points of $\mathbb{R}^3$ are solutions of the given system.

***E** The system has no solution.

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10 Consider the vector $(-3, 2, 4)$ in the basis $[\mathbf{u}, \mathbf{v}, \mathbf{w}]$, where $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Find its coordinates in the basis $[\mathbf{w}, \mathbf{u}, 2\mathbf{v}]$.

A $(-3, 2, 8)$

B $(4, -3, 4)$

C $(-3, 2, 4)$

D $(-3, 2, 2)$

***E** $(4, -3, 1)$

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11 Which of the following maps of the type $\mathbb{R}^2 \to \mathbb{R}$ is linear?

A $f(x, y) = |2x + 3y|$

***B** $f(x, y) = 2^2 x + 3^2 y$

C $f(x, y) = (2x + 3y)^2$

D $f(x, y) = 2xy$

E $f(x, y) = 2x^2 + 3y^2$

Calculus

12 Compute the value of the integral $\int_0^\pi x \sin x \, \mathrm{d}x$.

A 0

B $\pi + 2$

***C** π

D $-\pi$

E $-\pi + 2$

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13 An excavator is moving along a straight line. Its position in the time $t \in [0, 10]$ is given as $f(t) = 2t^3 - 21t^2 + 60t - 41$.
Find the least time $t \in (0, 10)$ when the excavator stops.

A The excavator does not stop at any time $t \in (0, 10)$.

B $t = 5$

***C** $t = 2$

D $t = \frac{19-\sqrt{33}}{4}$

E $t = 1$

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14 The function $f\colon \mathbb{R} \to \mathbb{R}$ given by the formula $f(x) = \sin(x^2)$ is:

A odd
B surjective
C injective
***D** even
E periodic

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15 Compute the value of the limit $\lim_{x\to 0} \frac{2^x-1}{x}$. (The function $\ln x$ is the natural logarithm of x.)

***A** $\ln 2$

B $\frac{1}{\ln 2}$

C ∞

D 1

E $\log_2 e$

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16 Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(0) = f(10) = 10$. Which of the following statements about f is generally valid?

A There exists $x_0 \in (0, 10)$ such that $f(x_0) = 10$.

B If f has a first derivative on the interval $[0, 10]$ then it has a global minimum on $[0, 10]$.

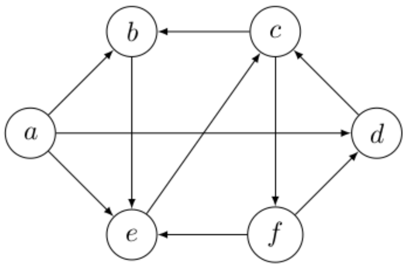
C There exists $x_0 \in \mathbb{R}$ such that f has a global extremum at x_0.

D The function $f(x)$ has a finite first derivative for all $x \in \mathbb{R}$.

***E** If f has a finite first derivative on the interval $[0, 10]$ then there exists $x_0 \in [0, 10]$ such that $f'(x_0) = 0$.

Graph theory

17 Consider the following directed graph:



Decide which of the following sequences **can be** a sequence in which elements are discovered during the **breadth-first traversal** of the given graph from the element a. (We do not suppose any particular ordering of vertices of the graph. The order in which the algorithm discovers new vertices is thus not unique.)

A a, b, c, d, e, f

***B** a, e, b, d, c, f

C a, b, d, e, f, c

D a, b, d, c, e, f

E a, b, e, c, f, d

18 What is the least number of edges that a **connected undirected graph** with n vertices can have for $n \geq 1$?

A $2n$

B 0

C 1

D n

***E** $n - 1$

19 What is the number of all non-isomorphic undirected graphs on 4 vertices?

A 9

***B** 11

C 10

D 8

E 12

20 Let G be an arbitrary directed weighted graph with non-negative weights of edges. For an arbitrary pair of vertices u and v of the graph G denote as $\delta(u, v)$ the length of a shortest path (with respect to the sum of edge weights) from the vertex u to the vertex v. If the path does not exist, let $\delta(u, v) = \infty$. Which of the following claims holds for arbitrary vertices u, v, w of the graph G?

A $\delta(u, w) \geq \delta(u, v) + \delta(v, w)$

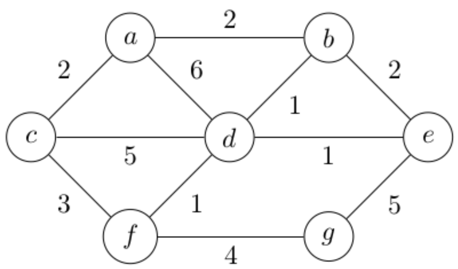
B $\delta(u, w) = \delta(u, v) + \delta(v, w)$

***C** $\delta(u, w) \leq \delta(u, v) + \delta(v, w)$

D $\delta(u, w) = \delta(u, v) + \delta(v, w) - 1$

E $\delta(u, w) = \delta(w, u)$

21 Consider the following undirected graph G with given weights of edges:



What is the weight (with respect to the sum of edge weights) of an arbitrary minimal spanning tree of the graph G?

A 13

B 5

C 9

D 7

***E** 11

Probability

22 Let us roll a die two times independently in succession. Compute the conditional probability of the second number rolled being 4, assuming that the first number rolled is smaller than the second one.

***A** $\frac{1}{5}$

B $\frac{1}{6}$

C $\frac{2}{5}$

D $\frac{1}{7}$

E $\frac{2}{7}$

23 Consider the data set $1, 1, 2, 2, 2, 4, 8, 8, x, x$. What is the value of x for which the mean of this data set is equal to 4?

***A** 6
B 4
C 5
D 3
E 7

24 Consider a random variable X being equal to the value of a random roll of a die. Compute the variance of the random variable $Y = 2X + 1$.

A $\frac{41}{6}$

***B** $\frac{35}{3}$

C $\frac{47}{6}$

D $\frac{35}{6}$

E $\frac{47}{12}$

25 During an exam, a student draws 2 questions out of 10. The probability of drawing each of the questions is the same. What is the probability that the student will draw at least one question that she cannot answer, if she can answer 7 of the questions.

A $\frac{3}{10}$

***B** $\frac{8}{15}$

C $\frac{2}{7}$

D $\frac{7}{15}$

E $\frac{1}{7}$